Section 5.1

Areas and Distances

(1) The Area Problem

(2) The Distance Problem

(3) Summation Notation

Area

Area is a measure of the size of 2-dimensional shapes.

Area is preserved under cutting, gluing, sliding, and rotating.

Area(A)=Area(B)+Area(C)

There are standard formulas for the areas of common shapes:

Rectangle: $A = bh$ **Triangle:** $A = \frac{1}{2}bh$ **Circle:** $A = \pi r^2$

But what about more complicated shapes?

The Area Problem

The motivation for this chapter is the problem of calculating the area of more general regions, such as the area under the graph of a function $y = f(x)$.

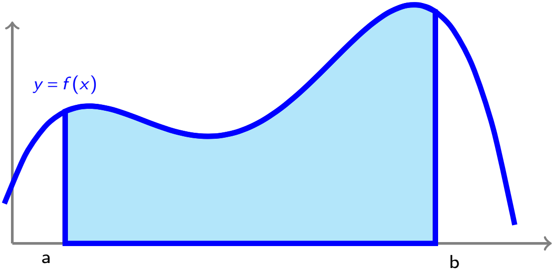
When we studied tangent lines, we soon discovered that we needed to use limits to calculate them in a mathematically rigorous way. This led to the concept of a derivative.

Similarly, calculating area in a rigorous way will also require limits and will lead us to a new mathematical concept: the **integral**.

The Area Problem

Let $f(x)$ be continuous and positive on a closed interval $[a, b]$.

What is the area of the region bounded by the graph of $f(x)$, the vertical lines $x = a$ and $x = b$, and the x-axis?

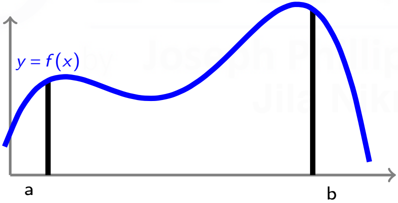


The Area Problem

The area A under the graph of f between $x = a$ and $x = b$ can be approximated as the total area of n rectangles.

- Divide the domain $[a, b]$ into n segments of length $\Delta x = \frac{b-a}{n}$.
- Inside each segment, choose a value x_i.
- Form a rectangle of height $f(x_i)$ on each segment.

Then $\boxed{A \approx f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x.}$



Example 1: Approximate the area under $y = x^2$ on $[1,4]$ using 6 segments. **Here $a = 1$, $b = 4$, $n = 6$, and $\Delta x = \frac{b-a}{n} = \frac{1}{2}$. The 6 segments of the domain are**

$[1,1.5]$ $[1.5,2]$ $[2,2.5]$ $[2.5,3]$ $[3,3.5]$ $[3.5,4]$

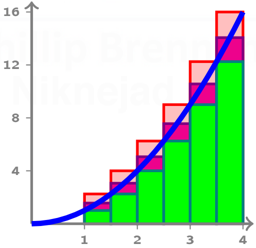
x_i = **right endpoint** of i^{th} **interval**:

i	x_i	$f(x_i)$	$f(x_i)\Delta x$
1	1.5	2.25	1.125
2	2	4	2
3	2.5	6.25	3.125
4	3	9	4.5
5	3.5	12.25	6.125
6	4	16	8
		Area ≈	**24.875**

Choice of x_i	Estimate of area
Right endpoints	**24.875 (too high)**
Left endpoints	**17.375 (too low)**
Midpoints	**20.9375 (closest)**

x_i = **left endpoint** of i^{th} **interval**:

i	x_i	$f(x_i)$	$f(x_i)\Delta x$
1	1	1	0.5
2	1.5	2.25	1.125
3	2	4	2
4	2.5	6.25	3.125
5	3	9	4.5
6	3.5	12.25	6.125
		Area ≈	**17.375**



x_i = **midpoint** of i^{th} **interval**:

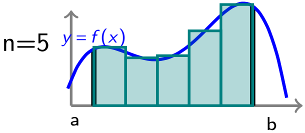
i	x_i	$f(x_i)$	$f(x_i)\Delta x$
1	1.25	1.5625	0.78125
2	1.75	3.0625	1.53125
3	2.25	5.0625	2.53125
4	2.75	7.5625	3.78125
5	3.25	10.5625	5.28125
6	3.75	14.0625	7.03125
		Area ≈	**20.9375**

Area Expressed as a Limit

The area A under the graph of f between $x = a$ and $x = b$ can be approximated as the total area of n rectangles:

$$A \approx \overbrace{f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x}^{R_n}$$

As n gets larger and larger, the approximation R_n gets better and better.



The exact area is given by a limit.

The **area** A under the graph of a continuous function f between $x = a$ and $x = b$ equals the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x\right)$$

Calculating Distance

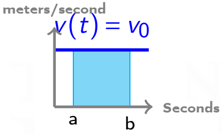
Let $v(t)$ be the velocity of an object at time t.

The area under the graph of $v(t)$ on a time interval $[a, b]$ measures the net distance traveled, or displacement, between times a and b.

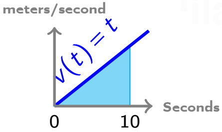
Note that the units:

$$\begin{aligned} \text{Units of area under the graph of } v(t) &= \text{units of } t \times \text{units of } v(t) \\ &= \text{time } \times \frac{\text{distance}}{\text{time}} \\ &= \text{distance.} \end{aligned}$$

Example 2(a): If $v = v_0$ on $[a, b]$, then the region under the graph is a rectangle with area $v_0(b - a)$.



Example 2(b): An object starts at rest and accelerates at a constant rate of 1 m/s^2 for 10 seconds. Then $v(t) = t$ m/s.
Displacement = area under the curve = $\frac{1}{2}(10\,\text{s})(10\,\text{m/s})$ = 50 m.



Example 3: You are driving across Missouri. In order to stay awake, you estimate how far you have traveled from your speedometer readings:

2:00 PM	70 mph (the speed limit)
2:15 PM	65 mph (up a small hill)
2:30 PM	75 mph (down the hill)
2:45 PM	55 mph (careful, is that a speed trap?)
3:00 PM	80 mph (vroom!)

You can now estimate[1] the maximum and minimum possible distance you have traveled during this hour:

Max: $\frac{1}{4}(70) + \frac{1}{4}(75) + \frac{1}{4}(75) + \frac{1}{4}(80) = 75$ miles
Min: $\frac{1}{4}(65) + \frac{1}{4}(65) + \frac{1}{4}(55) + \frac{1}{4}(55) = 60$ miles

The actual distance traveled is somewhere between these two estimates.

[1] Assuming that in each 15-minute interval, your max and min speeds occur at the endpoints.

Summation Notation

We have encountered expressions like

$$f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

that are sums of many similar-looking terms. We need a notation for writing sums in a compact form.

Summation Notation

The notation $\sum_{j=m}^{n} a_j$ means $a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$.

- $\sum$ is the Greek letter Sigma (mnemonic for "sum.")
- The notation $\sum_{j=m}^{n}$ tells us to start at $j = m$ and to end at $j = n$.
- a_j is called the **general term** and j is the **summation index**.

Summation Notation

The notation $\sum_{j=m}^{n} a_j$ means $a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$.

Examples:

$$\sum_{j=1}^{100} j = 1 + 2 + 3 + \ldots + 100$$

$$\sum_{j=4}^{785} j^2 = 4^2 + 5^2 + \ldots + 785^2$$

$$\sum_{j=4}^{6} (j^3 - j - 1) = (4^3 - 4 - 1) + (5^3 - 5 - 1) + (6^3 - 6 - 1)$$

Summation Notation and Area

Summation Notation

The notation $\sum_{j=m}^{n} a_j$ means $a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$.

Therefore, our estimate for the area under the graph of a continuous, positive function $f(x)$ on an interval $[a, b]$ is

$$R_n = f(x_1)\Delta x + \ldots + f(x_n)\Delta x = \sum_{j=1}^{n} f(x_j)\Delta x$$

and the exact area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1)\Delta x + \ldots + f(x_n)\Delta x)$$
$$= \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x.$$

Properties of Summations

If you understand addition, you understand summation!

$$\sum_{j=m}^{n} (a_j \pm b_j) = \left(\sum_{j=m}^{n} a_j\right) \pm \left(\sum_{j=m}^{n} b_j\right)$$

$$\sum_{j=m}^{n} (ca_j) = c \sum_{j=m}^{n} a_j \qquad \text{(for any constant } c\text{)}$$

$$\sum_{j=m}^{n} c = c(n - m + 1)$$

Properties of Summations

For example:

$$\sum_{j=1}^{1000}(3j^2-5j+3) = \left(\sum_{j=1}^{1000} 3j^2\right) - \left(\sum_{j=1}^{1000} 5j\right) + \left(\sum_{j=1}^{1000} 3\right)$$
$$= 3\left(\sum_{j=1}^{1000} j^2\right) - 5\left(\sum_{j=1}^{1000} j\right) + 3000$$

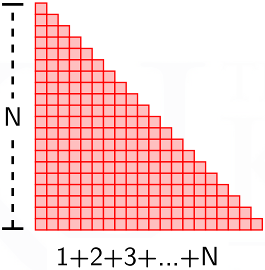
Fortunately, there are nice formulas for the sum of the first n numbers, squares, cubes, fourth powers, ...

Summation Formulas

- $\sum_{j=1}^{n} j = 1 + 2 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$
- $\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + \ldots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$

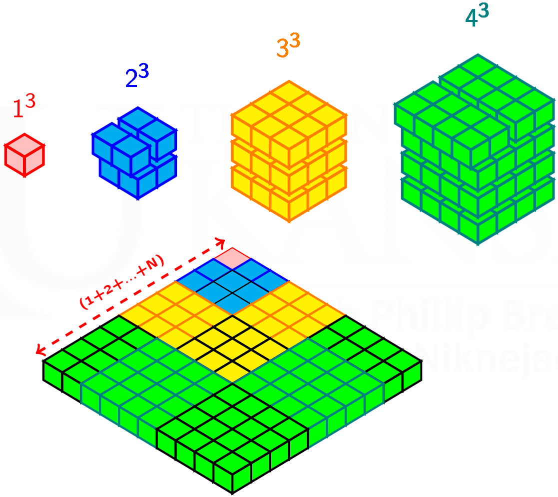
You don't have to memorize these formulas, but the first one has a very elegant explanation!

The Sum of the First N Integers



$$1 + 2 + \ldots + N = \frac{N \times (N+1)}{2}$$

The Sum of the Cubes of the First N Integers



$$1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$$